Rigorous proof of the attractive nature for the Casimir force of a *p*-odd hypercube

Xin-zhou Li and Xiang-hua Zhai

Department of Physics, Shanghai Normal University, Shanghai 200234, People's Republic of China

E-mail: kychz@shtu.edu.cn

Abstract
The Casimir effect giving rise to an attractive force between the configuration boundaries that confine the massless scalar field is rigorously proved for an odd dimensional hypercube with the Dirichlet boundary conditions and different spacetime dimensions D by the Epstein zeta function regularization.

PACS numbers: 04.62.+v, 03.65.Ge

In 1948 Casimir calculated an extraordinary property that two uncharged metallic plates would have an attractive force in vacuum [1]. This force is a strong function of a and is measurable only for $a < 1 \ \mu$m [2]. Boyer [3], who numerically calculated the Casimir force of a thin spherical shell, found that the sphere tends to be expanded. By using the mean approximation, the stresses are directed outwards for a cubic cavity, which tend to expand the cavity [4]. Therefore, one may imagine the spherical shell to be deformed into a cubic shell, and expect that this deformation does not change the resulting stresses. It is interesting to note that the Casimir force of a massless scalar field may be repulsive for p-odd cavity with unequal edges [5]. On the other hand, few physicists would nowadays argue against the statement that the zeta function regularization procedure has proven to be a very powerful and elegant technique. Rigorous extension of the proof of the Epstein zeta function regularization has been obtained [6]. The generalized zeta function has many interesting applications, e.g., in the piecewise string [7] and branes [8]. In this paper, we rigorously prove that the Casimir force is attractive for a p-odd hypercube with the Dirichlet boundary conditions and spacetime dimension D less than critical value D_c.

A Hermitian massless scalar field $\phi(t, x^a, x^T)$ is confined to the interior of $(D - 1)$-dimensional rectangular cavity Ω with p edges of finite lengths $L_1, L_2, \ldots, L_p$ and $D - 1 - p$ edges with characteristic lengths of order $\lambda \gg L_a$ where $i = 1, \ldots, D - 1$; $a = 1, \ldots, p$; $T = p + 1, \ldots, D - 1$. We consider the case of Dirichlet boundary conditions, i.e., $\phi(t, x^a, x^T)|_{\partial\Omega} = 0$. The Casimir energy density is [5]

$$\varepsilon_p^D(L_1, L_2, \ldots, L_p) = -\frac{\pi^{(D-p)/2}}{2^{D-p+1}} \Gamma\left(-\frac{D-p}{2}\right) E_p\left(\frac{1}{L_1^2}, \frac{1}{L_2^2}, \ldots, \frac{1}{L_p^2}; -\frac{D-p}{2}\right) \quad (1)$$

where the Epstein zeta function $E_p(a_1, a_2, \ldots, a_p; s)$ is defined as

$$E_p(a_1, a_2, \ldots, a_p; s) = \sum_{\{n\}=1}^{\infty} \left(\sum_{j=1}^{p} a_j n_j^2 \right)^{-s}. \tag{2}$$

Assuming $L_1 = L_2 = \cdots = L_p = L$, equation (1) can be reduced to

$$\varepsilon_p^D(L) = \frac{L^{p-D}}{2^{D+1}} \sum_{q=0}^{p-1} (-1)^{q+1} C_p^q \left(\sqrt{\pi} \right)^{q-D} \Gamma \left(\frac{D-q}{2} \right) A \left(1, \ldots, 1; \frac{D-q}{2} \right) \tag{3}$$

where the Epstein zeta function $A(a_1, a_2, \ldots, a_p; s)$ is defined as

$$A(a_1, a_2, \ldots, a_p; s) = \sum_{\{n\}=\infty}^{\infty}{}' \left(\sum_{j-1}^{p} a_j n_j^2 \right)^{-s} \tag{4}$$

where the prime means that the term $n_1 = n_2 = \cdots = n_p = 0$ has to be excluded.

Using equations (2)–(4) and the Mellin transformation of $\exp(-b\tau)$

$$\int_0^{\infty} \tau^{s-1} e^{-b\tau} \, d\tau = b^{-s} \Gamma(s) \tag{5}$$

we have

$$\begin{aligned}
\varepsilon_p^D(L_1, L_2, \ldots, L_p) &= L_p \varepsilon_{p-1}^D(L_1, L_2, \ldots, L_{p-1}) \\
&+ \frac{\pi^{(D-p)/2}}{2^{D-p+2}} \Gamma\left(-\frac{D-p}{2}\right) E_{p-1}\left(\frac{1}{L_1^2}, \ldots, \frac{1}{L_{p-1}^2}; -\frac{D-p}{2}\right) \\
&- \frac{1}{2^{D-p+1}} L_p^{-\frac{D-p}{2}} \sum_{k=0}^{\infty} \frac{(16\pi)^{-k}}{k!} L_p^{-k} \prod_{j=1}^{k} [(D-p+1)^2 - (2j-1)^2] \\
&\times \sum_{\{n\}=1}^{\infty} n_p^{(p-D-2-2k)/2} \left(\frac{n_1^2}{L_1^2} + \cdots + \frac{n_{p-1}^2}{L_{p-1}^2} \right)^{\frac{D-p-2k}{4}} \\
&\times \exp\left[-2\pi L_p n_p \left(\frac{n_1^2}{L_1^2} + \cdots + \frac{n_{p-1}^2}{L_{p-1}^2} \right)^{\frac{1}{2}} \right]
\end{aligned} \tag{6}$$

and

$$\varepsilon_p^D(L) = \frac{L^{p-D}}{2^{D+1}} \pi^{-D/2} \int_0^{\infty} d\tau \, (\sqrt{\tau})^{D-2} \left\{ \left[1 - \left[\frac{\pi}{\tau}\right]^{\frac{1}{2}}\right]^p - \left[\theta_3(0, e^{-\tau}) - \left[\frac{\pi}{\tau}\right]^{\frac{1}{2}}\right]^p \right\} \tag{7}$$

where the elliptic θ function

$$\theta_3(0, q) \equiv \sum_{m=-\infty}^{\infty} q^{m^2}. \tag{8}$$

When $p = 2j + 1p$ (j is a positive integer), it is obvious that $\varepsilon_p^D(L) < 0$ for any D, since $\theta_3(0, e^{-\tau}) > 1$ and integrand is always negative between the integration limits. On the other hand, numerical calculations show that the energy density is positive when p is even and $D \leqslant 6$ in the $p = 2j$ case [9]. Note that the terminology 'Casimir force' for L_i direction is in proportion to the derivative of Casimir energy with respect to L_i. We need to study firstly the behaviour of Casimir energy so as to discuss the nature of Casimir force. We can prove the following lemmata for the Casimir energy.

Lemma 1. *For two spacetime dimensions D_1 and D_2, if $D_2 > D_1$, and $\varepsilon_{2j}^{D_1}(L) \leqslant 0$, then $\varepsilon_{2j}^{D_2} < 0$.*

Proof. Define

$$k(\tau) \equiv \left[1 - \left(\frac{\pi}{\tau}\right)^{\frac{1}{2}}\right]^{2j} - \left[\theta_3\left(0, \, \mathrm{e}^{-\tau}\right) - \left(\frac{\pi}{\tau}\right)^{\frac{1}{2}}\right]^{2j} \tag{9}$$

which has only one real root $0 < \tau_0 < \pi$ for any j. Since $k(\tau) > 0$ for $0 < \tau < \tau_0$ and $k(\tau) < 0$ for $\tau_0 < \tau < \infty$, we have

$$\varepsilon_{2j}^{D_2}(L) = \frac{L^{p-D_2}}{2^{D_2+1}}\pi^{-D_2/2}\left(\int_0^{\tau_0} + \int_{\tau_0}^{\infty}\right) \mathrm{d}\tau\,(\sqrt{\tau})^{D_1+(D_2-D_1)-2}k(\tau)$$

$$< \left(\frac{1}{2L}\sqrt{\frac{\tau_0}{\pi}}\right)^{D_2-D_1}\varepsilon_{2j}^{D_1}(L) \leqslant 0. \tag{10}$$

$\square$

Lemma 2. *There exists a particular spacetime dimension $D_c = 6$, such that for $D \leqslant D_c$, $\varepsilon_2^D(L) > 0$ and for $D > D_c$, $\varepsilon_2^D(L) < 0$.*

Proof. In the $p = 2$ case, equation (7) can be reduced to

$$\varepsilon_2^D(L) = \frac{L^{2-D}}{2^{D-1}\pi^{D/2}}\left[2\sqrt{\pi}\Gamma\left(\frac{D-1}{2}\right)\zeta\left(\frac{D-1}{2}\right)\beta\left(\frac{D-1}{2}\right) - \Gamma\left(\frac{D}{2}\right)\zeta\left(\frac{D}{2}\right)\beta\left(\frac{D}{2}\right)\right] \tag{11}$$

where $\zeta(r)$ is the Riemann zeta function and $\beta(r)$ is the Dirichlet series

$$\beta(r) \equiv \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)^r}. \tag{12}$$

Using these known one-dimensional series, we have $\varepsilon_2^3(L) = 0.04104$, $\varepsilon_2^4(L) = 0.00483$, $\varepsilon_2^5(L) = 0.00081$, $\varepsilon_2^6(L) = 0.00011$ and $\varepsilon_2^7(L) = -1.9 \times 10^{-5}$ if L is the chosen unit length. Thus, we show lemma 2 from lemma 1. $\square$

Lemma 3. *If j is large enough, then $\varepsilon_{2j}^D(L) < 0$.*

Proof. For $\tau > 4\pi > \tau_0$, one can easily find

$$|k(\tau)| > \frac{p\mathrm{e}^{-\tau}}{2^{2j-2}}. \tag{13}$$

From equation (7), we have

$$\varepsilon_{2j}^D(L) < \frac{L^{2j-D}}{2^{D+1}}\pi^{-\frac{D}{2}}\left(2\pi^j\tau_0^{\frac{1}{2}} - \int_{64\pi}^{\infty} \mathrm{d}\tau\,\tau^{(p-1)/2}|k(\tau)|\right)$$

$$< 2^D\left(4\sqrt{\pi}L\right)^{2j-D}\left[4^{-2j}\tau_0^{\frac{1}{2}} - p(4\sqrt{\pi})^{-1}\mathrm{e}^{-64\pi}\right]. \tag{14}$$

Then, for j large enough, $\varepsilon_{2j}^D(L) < 0$. $\square$

Lemma 4. *If D is large enough, then $\varepsilon_{2j}^D(L) < 0$ for any j.*

Proof. From equation (7), we have

$$\varepsilon_{2j}^D(L) < \frac{L^{2j-D}}{2^{D+1}} \pi^{-\frac{D}{2}} \left(\int_0^{\tau_0} + \int_{\tau_0}^\infty \right) d\tau \, (\sqrt{\tau})^{D-2} k(\tau)$$

$$< \frac{L^{2j-D} \tau_0^{\frac{D}{2}-1}}{2^{\frac{D}{2}+2} \pi^{\frac{D}{2}}} \left[\frac{1}{2^{\frac{D}{2}-1}} \int_0^{\tau_0} d\tau \, k(\tau) + \int_{\tau_0}^\infty d\tau \, k(\tau) \right]. \tag{15}$$

Since the first term is positive and the second is negative in the brackets of equation (15), when D is large enough, $\varepsilon_{2j}^D(L) < 0$ for any j. $\qquad\square$

Lemma 5. *The Casimir force of a p-dimensional rectangular cavity with the Dirichlet boundary conditions can be written in terms of the $(p-1)$-dimensional Casimir energy density and multiseries with exponential factors.*

Proof. From the recursion relation of the Casimir energy (density equation (6)), per unit area

$$-\frac{\partial \varepsilon_p^D(L_1, \ldots, L_p)}{\partial L_p} = -\varepsilon_{p-1}^D(L_1, \ldots, L_{p-1})$$

$$+ \frac{1}{2^{D-p+2}} \sum_{k=0}^\infty \frac{(16\pi)^{-k}}{k!} (p-D-2k) L_p^{\frac{p-D-2k-2}{2}} \prod_{j=1}^k [(D-p+1)^2 - (2j-1)^2]$$

$$\times \sum_{\{n\}=1}^\infty n_p^{\frac{p-D+2+2k}{2}} \left(\frac{n_1^2}{L_1^2} + \cdots + \frac{n_{p-1}^2}{L_{p-1}^2} \right)^{\frac{D-p-2k}{4}}$$

$$\times \exp\left[-2\pi L_p n_p \left(\frac{n_1^2}{L_1^2} + \cdots + \frac{n_{p-1}^2}{L_{p-1}^2} \right)^{\frac{1}{2}} \right]$$

$$- \frac{1}{2^{D-p}} \sum_{k=0}^\infty \frac{(16\pi)^{1-k}}{k!} L_p^{\frac{p-D-2k}{2}} \prod_{j=1}^k [(D-p+1)^2 - (2j-1)^2]$$

$$\times \sum_{\{n\}=1}^\infty n_p^{\frac{p-D+4+2k}{2}} \left(\frac{n_1^2}{L_1^2} + \cdots + \frac{n_{p-1}^2}{L_{p-1}^2} \right)^{\frac{D-p-2k+2}{4}}$$

$$\times \exp\left[-2\pi L_p n_p \left(\frac{n_1^2}{L_1^2} + \cdots + \frac{n_{p-1}^2}{L_{p-1}^2} \right)^{\frac{1}{2}} \right]. \tag{16}$$

This completes the proof of lemma 5. $\qquad\square$

Since the second and the third terms are obviously negative on the right-hand side of equation (16), the pressure is always negative (directed inwards) if the first term is also negative. From lemmata 1–4, we show that there exists a critical value of the spacetime dimension D_c for which $\varepsilon_{2j}^D > 0$ if $D \leqslant D_c$ and $\varepsilon_{2j}^D < 0$ if $D > D_c$. However, $2j + 1 > D_c$ if $j > 14$. In this case there is no critical D_c since all ε_{2j}^D (for $D = 2j+1, 2j+2, \ldots$) are negative. We summarize the above in the following.

Theorem 1. *The Casimir effect gives rise to an attractive force between the configuration boundaries that confine the massless scalar field for a p-odd hypercube with the Dirichlet boundary conditions and $D \leqslant D_c, p \leqslant 29$.*

Table 1. The critical value D_c for massless scalar fields satisfying Dirichlet boundary conditions inside a hypercube with *p*-odd unit sides in a *D*-dimensional spacetime.

p	3	5	7	9	11	13	15	17	19	21	23	25	27	29
D_c	7	9	11	12	14	16	17	19	21	23	24	26	28	30

Furthermore, we show the dependence of the critical value D_c on p in table 1 using numerical calculation for possible physical application.

Finally, we shall give a brief discussion on our result. In spite of an impressive literature on the Casimir effect [10], the query whether its attractive or repulsive character changes by going to higher dimensions had never been elucidated for Dirichlet boundary conditions. We analytically show that Casimir force of a *p*-odd hypercube is attractive, in contrast to the result of [11]. Our result is consistent with numerical calculation [9]. It may be worth emphasizing that the Epstein zeta function is the fundamental zeta function associated with higher dimensions.

Acknowledgments

References

[1] Casimir H B G 1948 *Proc. K. Ned. Akad. Wet.* **51** 793
[2] Sparmary M Y 1958 *Physica* **24** 751
      Mohideen U and Roy A 1998 *Phys. Rev. Lett.* **81** 4549
[3] Boyer T H 1968 *Phys. Rev.* **174** 1764
[4] Lukosz W 1971 *Physica* **56** 109
[5] Li X Z, Cheng H B, Li J M and Zhai H X 1997 *Phys. Rev.* D **56** 2155
[6] Elizald E and Romeo A 1989 *Phys. Rev.* D **40** 436
      Elizald E and Romeo A 1989 *Rev. Math. Phys.* **1** 113
[7] Li X Z, Shi X and Zhang J 1991 *Phys. Rev.* D **44** 560
[8] Shi X and Li X Z 1991 *Class. Quantum Grav.* **8** 75
[9] Ambjφrn J and Wolfram S 1983 *Ann. Phys., NY* **147** 1
[10] Plunien G, Muller B and Greiner W 1986 *Phys. Rep.* **134** 87
[11] Caruso F, Neto N P, Svaiter B F and Svaiter N F 1991 *Phys. Rev.* D **43** 1300